MATH 642:491

Problem Set 4. Elementary number theory [1]

Problems marked with (*) are more interesting/challenging. The last few are Putnam problems.

1. Prove that for all integers $n \geq 1$, $b_n = \sum_{i=0}^{n} 10^i$ is a not perfect square.
2. Prove: the product of side lengths of a right triangle with integer side lengths is divisible by 60.
3. Determine all integers n such that $n^4 - n^2 + 64$ is a perfect square.
4. Show that for every composite integer n there are positive integers x, y, z so that $xy + xz + yz + 1 = n$.
5. Suppose x can be be written as a sum of two squares of integers, and y can be written as a sum of two squares of integers. Show that xy can also be written as a sum of two squares of integers.
6. Prove that every positive integer has a multiple whose decimal representation contains all 10 digits.
7. For any integer $n \geq 0$, the nth Fibonacci number is defined via the following recurrence: $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 1$ and $F_1 = 1$. Prove that any two successive Fibonacci numbers are relatively prime. (Two integers are relatively prime if they have no common factor other than 1).
8. Show that $1^{2023} + 2^{2023} + \ldots + 2024^{2023}$ is a multiple of 2025.
9. Show that if $2n + 1$ and $3n + 1$ are both perfect squares then n is divisible by 40.
10. (*) Prove that every integer is a divisor of infinitely many Fibonacci numbers.
11. (*) For any positive natural number n, let $H(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$. Prove that for $n \geq 2$, $H(n)$ is not an integer.
12. (*) Say that a positive integer is *alternating* if when expressed in base 2, the digits alternate 0,1. Prove that the number of alternating primes is finite.
13. (*) Let $S(n)$ be the number of positive integer solutions (x, y) to the equation $1/x + 1/y = 1/n$. Determine the size of $S(n)$.
14. (*Putnam) For positive integers m, n, let

$$f(m, n) = \sum_{i=0}^{mn-1} (-1)^{\lfloor \frac{i}{m} \rfloor + \lfloor \frac{i}{n} \rfloor}.$$

For what pairs (m, n) is $f(m, n) = 0$?

[1] Version 6-30-2025

15. (*Putnam) Let p be a prime and let $\mathbb{F}_p$ be the field of integers $\mod p$. How large is the set $\{x^2 : x \in \mathbb{F}_p\} \cap \{y^2 + 1 : y \in \mathbb{F}_p\}$.

16. (*Putnam) Let $(a_n : n \geq 1)$ be a sequence given by $a_1 = 3$ and for $i > 1$, and $a_i = 3^{a_{i-1}}$. What two digit numbers between 00 and 99 appear as the last two digits of infinitely many terms a_n.